Int. J. Solids Structures, 1966, Vol. 2, pp. 9 to 26. Pergamon Press Ltd. Printed in Great Britain

EXTENDED DISPLACEMENT BOUND THEOREMS FOR WORK HARDENING CONTINUA SUBJECTED TO IMPULSIVE LOADING*

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Abstract—Earlier work presenting a technique for bounding displacements following impulsive loading in inelastic continua is extended to give more powerful results for stable work hardening materials. The extension is achieved by adding a self-evident statement regarding the material behavior to Drucker's postulate of stability for time independent, path dependent solids.

1. INTRODUCTION

IN A PREVIOUS PAPER [1] a general principle relating independent equilibrium stress fields and compatible strain fields was developed for a path-dependent work hardening solid. In conformity with many principles of this type, the strains considered are not necessarily infinitesimally small (*small–small*), but nevertheless sufficiently small that the effect of the strain on the undeformed geometry of the solid may be ignored (*large–small*).

The principle was developed for a class of work hardening materials which satisfies Drucker's postulated criterion of stability [2] for *large–small* deformations. It was found, however, that the applications of the principle were limited by a necessary requirement that there should exist a substantial range of path independent or elastic behavior. This paper is motivated by a desire to remove this limitation and to develop a more powerful principle for large–small deformations.

In order to achieve this extension we shall make use of an additional statement about the material which introduces no further postulates. It will then be shown that a more general principle may be developed which in certain cases can be usefully applied to the computation of displacement bounds for impulsively loaded structures and continua.

We shall consider initially a small homogeneously stressed element of the material, and assume that changes in stress are imposed on the material. The state of stress may be represented by a stress point in a nine-dimensional orthogonal stress space: The nine coordinates of the stress point are then the nine components of the stress tensor σ_{ij}. A stress path between two states σ_{ij}^a and σ_{ij}^b is the locus of the stress point between the initial and final points. The stress–strain relation will map a strain path corresponding to the change in stress. The material may be path dependent: thus changes in strain which occur in changing the stress state may depend on the stress path.

* The results presented in this paper were obtained in the course of research sponsored by the David Taylor Model Basin, Underwater Explosions Research Division, under Contract N189(181)-57827A(X).

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All materials considered in his paper will be assumed time independent. This means that, in following a given stress path between two states, the stress rate $\dot{\sigma}_{ij}$ is of no consequence in determining the strain response.

2. STABILITY POSTULATES

We shall consider only materials which satisfy the following postulated properties for large–small deformations:

(i) If the stress path between two stress states σ_{ij}^a, σ_{ij}^b is a straight line in stress space,

$$\int_{\sigma_{ij}^a}^{\sigma_{ij}^b} (\varepsilon_{ij}^* - \varepsilon_{ij}^a)\, d\sigma_{ij}^* \geq 0. \tag{1}$$

The asterisk will be used to denote a straight line path. It must be emphasized that a straight line path is taken to indicate a monotonic change in $|\sigma_{ij} - \sigma_{ij}^a|$ (Fig. 1). The strain ε_{ij}^a is associated with σ_{ij}^a, and is dependent on the entire previous stress history. ε_{ij}^a remains constant in the integrand of (1), and thus $(\varepsilon_{ij}^* - \varepsilon_{ij}^a)$ indicates changes in strain measured from ε_{ij}^a.

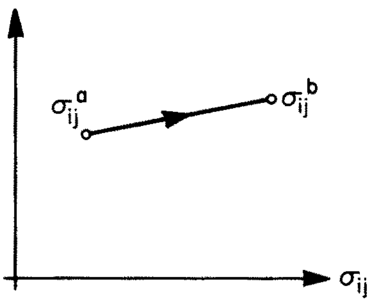

FIG. 1.

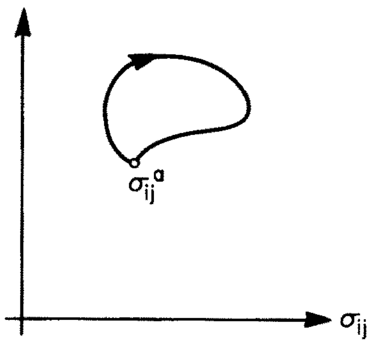

FIG. 2.

(ii) The net complementary work integral around any cycle in stress space (Fig. 2) is negative or zero.

$$\oint (\varepsilon_{ij} - \varepsilon_{ij}^a)\, d\sigma_{ij} \leq 0. \tag{2}$$

These postulates are identical to those given by Drucker [2]. Inequality (1) follows directly from the postulate for small–small changes, usually written in the form

$$\Delta\sigma_{ij}\Delta\varepsilon_{ij} \geq 0 \tag{3}$$

where $\Delta\varepsilon_{ij}$ is the change in strain produced by an infinitesimally small change in stress $\Delta\sigma_{ij}$ (cf. Drucker [3]). Inequality (2) is equivalent to the postulate that the net work integral around any cycle is non-negative.

It may be noted that by considering any path from σ_{ij}^a to σ_{ij}^b (Fig. 3) and then assuming that the cycle is completed by a straight line path from σ_{ij}^b to σ_{ij}^a we may derive an inquality for net work along any path. For the path given in Fig. 3, from (2),

$$\begin{aligned}&\int_{\sigma_{ij}^a}^{\sigma_{ij}^b}(\varepsilon_{ij}-\varepsilon_{ij}^a)\,\mathrm{d}\sigma_{ij}+\int_{\sigma_{ij}^b}^{\sigma_{ij}^a}(\varepsilon_{ij}^*-\varepsilon_{ij}^a)\,\mathrm{d}\sigma_{ij}^*\\&=\int_{\sigma_{ij}^a}^{\sigma_{ij}^b}(\varepsilon_{ij}-\varepsilon_{ij}^a)\,\mathrm{d}\sigma_{ij}+\int_{\sigma_{ij}^b}^{\sigma_{ij}^a}(\varepsilon_{ij}^*-\varepsilon_{ij}^b)\,\mathrm{d}\sigma_{ij}^*+(\varepsilon_{ij}^b-\varepsilon_{ij}^a)(\sigma_{ij}^a-\sigma_{ij}^b)\le 0\end{aligned}\tag{4}$$

where, as before, the asterisk indicates a straight line path. Using the fact that

$$\int_{\sigma_{ij}^a}^{\sigma_{ij}^b}(\varepsilon_{ij}-\varepsilon_{ij}^a)\,\mathrm{d}\sigma_{ij}+\int_{\varepsilon_{ij}^a}^{\varepsilon_{ij}^b}(\sigma_{ij}-\sigma_{ij}^a)\,\mathrm{d}\varepsilon_{ij}=(\sigma_{ij}^b-\sigma_{ij}^b)(\varepsilon_{ij}^b-\varepsilon_{ij}^a)\tag{5}$$

and inequality (1), (4) becomes

$$\int_{\varepsilon_{ij}^a}^{\varepsilon_{ij}^b}(\sigma_{ij}-\sigma_{ij}^a)\,\mathrm{d}\varepsilon_{ij}\ge\int_{\sigma_{ij}^b}^{\sigma_{ij}^a}(\varepsilon_{ij}^*-\varepsilon_{ij}^b)\,\mathrm{d}\sigma_{ij}^*\ge 0.\tag{6}$$

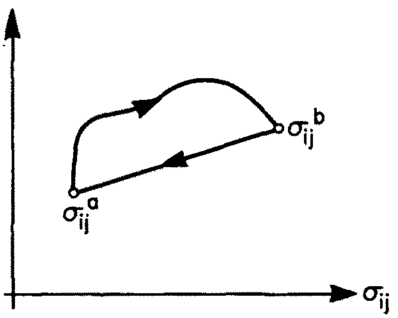

FIG. 3.

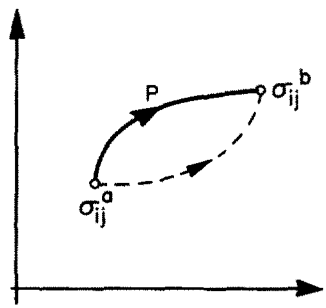

FIG. 4.

This form of Drucker's postulates was used in developing the earlier principle [1].

We shall use the following self-evident statement in addition to the postulates (1) and (2): of all physically possible paths from σ_{ij}^a to σ_{ij}^b there exists a path p such that

$$\int_{\sigma_{ij}^a}^{\sigma_{ij}^b}(\varepsilon_{ij}^p-\varepsilon_{ij}^a)\,\mathrm{d}\sigma_{ij}^p\ge\int_{\sigma_{ij}^a}^{\sigma_{ij}^b}(\varepsilon_{ij}-\varepsilon_{ij}^a)\,\mathrm{d}\sigma_{ij}.\tag{7}$$

Thus p is a path (or a member of a family of paths) for which the net complementary work done in changing the state of stress from σ_{ij}^a to σ_{ij}^b is a maximum. It is obvious from (1) that this maximum complementary work is non-negative. We may expect, for reasonable materials, that the maximum complementary work for a given stress change is finite, but this is not a necessary restriction from a purely mathematical point of view.

In the following section we shall use this maximum complementary work path to establish a general principle for arbitrary equilibrium stress fields and compatible strain fields in a continuum. It will become evident that this principle cannot be used in practice unless the p path is known. In consequence, we shall discuss some simple cases where the maximum work path can be found.

3. THE GENERAL PRINCIPLE

Consider an element of material with stress σ^a_{ij} and strain ε^a_{ij}. We suppose that there may occur two independent stress changes (Fig. 5). First, let the stress be changed from σ^a_{ij} to σ^s_{ij} by a p path, i.e. a path for which the net complementary work is a maximum. Second let the stress be changed from σ^a_{ij} to σ^c_{ij} by some unspecified path, and consider a further change in stress from σ^c_{ij} to σ^s_{ij} by a straight line path. Then

$$\int_{\sigma^a_{ij}}^{\sigma^s_{ij}} (\varepsilon^p_{ij}-\varepsilon^a_{ij})\,\mathrm{d}\sigma^p_{ij} \geq \int_{\sigma^a_{ij}}^{\sigma^c_{ij}} (\varepsilon_{ij}-\varepsilon^a_{ij})\,\mathrm{d}\sigma_{ij} + \int_{\sigma^c_{ij}}^{\sigma^s_{ij}} (\varepsilon^*_{ij}-\varepsilon^a_{ij})\,\mathrm{d}\sigma^*_{ij}. \tag{8}$$

Now

$$\int_{\sigma^c_{ij}}^{\sigma^s_{ij}} (\varepsilon^*_{ij}-\varepsilon^a_{ij})\,\mathrm{d}\sigma^*_{ij} = \int_{\sigma^c_{ij}}^{\sigma^s_{ij}} (\varepsilon^*_{ij}-\varepsilon^c_{ij})\,\mathrm{d}\sigma^*_{ij} + (\varepsilon^c_{ij}-\varepsilon^a_{ij})(\sigma^s_{ij}-\sigma^c_{ij}) \tag{9}$$

and

$$\int_{\sigma^a_{ij}}^{\sigma^c_{ij}} (\varepsilon_{ij}-\varepsilon^a_{ij})\,\mathrm{d}\sigma_{ij} = (\sigma^c_{ij}-\sigma^a_{ij})(\varepsilon^c_{ij}-\varepsilon^a_{ij}) - \int_{\sigma^a_{ij}}^{\varepsilon^c_{ij}} (\sigma_{ij}-\sigma^a_{ij})\,\mathrm{d}\varepsilon_{ij}. \tag{10}$$

From (8), (9), (10) and (1),

$$\int_{\sigma^a_{ij}}^{\sigma^s_{ij}} (\varepsilon^p_{ij}-\varepsilon^a_{ij})\,\mathrm{d}\sigma^p_{ij} + \int_{\varepsilon^a_{ij}}^{\varepsilon^c_{ij}} (\sigma_{ij}-\sigma^a_{ij})\,\mathrm{d}\varepsilon_{ij} - (\sigma^s_{ij}-\sigma^a_{ij})(\varepsilon^c_{ij}-\varepsilon^a_{ij}) \geq \int_{\sigma^c_{ij}}^{\sigma^s_{ij}} (\varepsilon^*_{ij}-\varepsilon^c_{ij})\,\mathrm{d}\sigma^*_{ij} \geq 0. \tag{11}$$

We now consider the entire continuum. Let surface tractions T^a_i, displacements u^a_i, stresses σ^a_{ij}, strains ε^a_{ij} represent the initial state of the body. Assume body forces are zero: body forces may be included (cf [1]) but are omitted for clarity.

Let σ^s_{ij} be a stress field which is in internal equilibrium, so that

$$\partial_j \sigma^s_{ij} = 0 \tag{12}$$

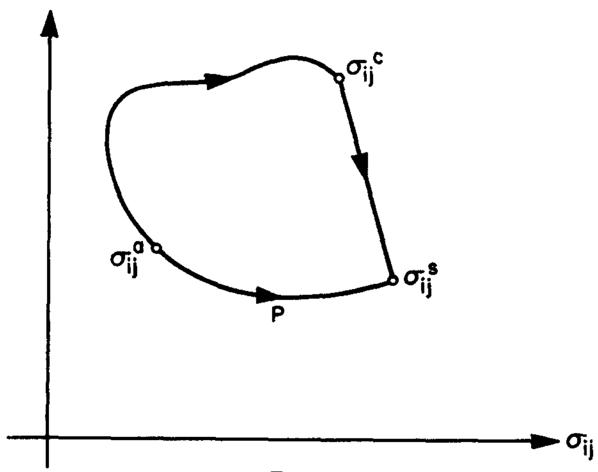

FIG. 5.

where ∂_j represents spatial differentiation, and in external equilibrium with body forces T_i^s,

$$T_i^s = \nu_j \sigma_{ij}^s, \tag{13}$$

ν_j is the unit outward normal at the point under consideration.

Let ε_{ij}^c be a strain field for the continuum which is compatible with displacement u_i^c. Thus if u_i^c is continuous and continuously differentiable

$$\varepsilon_{ij}^c = \tfrac{1}{2}(\partial_j u_i^c + \partial_i u_j^c). \tag{14}$$

We may now write, by the principle of virtual work,

$$\int_A (T_i^s - T_i^a)(u_i^c - u_i^a)\,\mathrm{d}A = \int_V (\sigma_{ij}^s - \sigma_{ij}^a)(\varepsilon_{ij}^c - \varepsilon_{ij}^a)\,\mathrm{d}V \tag{15}$$

where A and V are respectively the surface area and volume of the continuum. Notice that it is not necessary that the assumed stress and strain fields should satisfy any particular boundary conditions.

We may now consider independent changes from the initial state: ε_{ij}^a changing to ε_{ij}^c along any unspecified path, σ_{ij}^a changing to σ_{ij}^s along a maximum complementary work path. Thus, integrating (11) over the volume, and substituting from (15),

$$\int_V \mathrm{d}V \int_{\sigma_{ij}^a}^{\sigma_{ij}^s} (\varepsilon_{ij}^p - \varepsilon_{ij}^a)\,\mathrm{d}\sigma_{ij}^p + \int_V \mathrm{d}V \int_{\varepsilon_{ij}^a}^{\varepsilon_{ij}^c} (\sigma_{ij} - \sigma_{ij}^a)\,\mathrm{d}\varepsilon_{ij} \geq \int_A (T_i^s - T_i^a)(u_i^c - u_i^a)\,\mathrm{d}A. \tag{16}$$

This expression is similar to that found in [1], except that the complementary work term on the left hand side was previously required to be taken over an elastic (i.e. reversible) path.

It will be usually extremely difficult to apply this new principle in practice since the p path will not be known and will probably depend on the state σ_{ij}^a. These are some simple cases where the p path can be found, and these will be discussed in the following section.

4. MAXIMUM COMPLEMENTARY WORK PATHS FOR SIMPLE CASES

In order to establish continuity with previous work [1], we shall consider first a material with a substantial range of elastic behavior. Figure 6 shows two independent paths from σ_{ij}^a to σ_{ij}^b, one of which lies entirely within the yield surface. The yield surface is that corresponding to σ_{ij}^a: all stress paths lying within the yield surface are reversible. Due to this reversibility, a path from σ_{ij}^b to σ_{ij}^a along the reversible path (to be denoted by the superscript r) and then back to σ_{ij}^b along the irreversible path is admissible. Hence from (2)

$$\int_{\sigma_{ij}^b}^{\sigma_{ij}^a} (\varepsilon_{ij}^r - \varepsilon_{ij}^b)\,\mathrm{d}\sigma_{ij}^r + \int_{\sigma_{ij}^a}^{\sigma_{ij}^b} (\varepsilon_{ij} - \varepsilon_{ij}^b)\,\mathrm{d}\sigma_{ij} \leq 0. \tag{17}$$

Now

$$\int_{\sigma_{ij}^b}^{\sigma_{ij}^a} (\varepsilon_{ij}^r - \varepsilon_{ij}^b)\,\mathrm{d}\sigma_{ij}^r = -\int_{\sigma_{ij}^a}^{\sigma_{ij}^b} (\varepsilon_{ij}^r - \varepsilon_{ij}^a)\,\mathrm{d}\sigma_{ij}^r + (\varepsilon_{ij}^a - \varepsilon_{ij}^b)(\sigma_{ij}^a - \sigma_{ij}^b) \tag{18}$$

and

$$\int_{\sigma_{ij}^a}^{\sigma_{ij}^b} (\varepsilon_{ij} - \varepsilon_{ij}^b)\,\mathrm{d}\sigma_{ij} = \int_{\sigma_{ij}^a}^{\sigma_{ij}^b} (\varepsilon_{ij} - \varepsilon_{ij}^a)\,\mathrm{d}\sigma_{ij} + (\varepsilon_{ij}^a - \varepsilon_{ij}^b)(\sigma_{ij}^b - \sigma_{ij}^a). \tag{19}$$

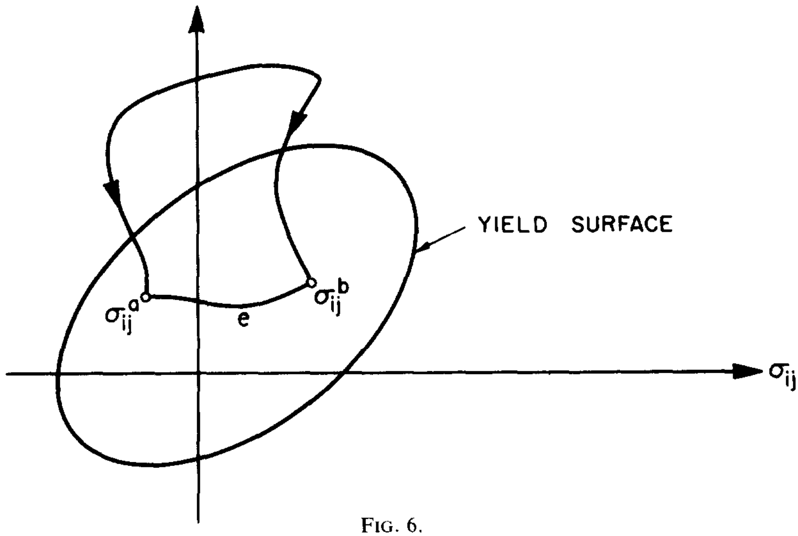

FIG. 6.

Hence from (17), (18) and (19)

$$\int_{\sigma_{ij}^a}^{\sigma_{ij}^b} (\varepsilon_{ij}-\varepsilon_{ij}^a)\,d\sigma_{ij} \le \int_{\sigma_{ij}^a}^{\sigma_{ij}^b} (\varepsilon_{ij}^r-\varepsilon_{ij}^a)\,d\sigma_{ij}^r. \tag{20}$$

Thus, if an elastic, reversible path exists between σ_{ij}^a and σ_{ij}^b, this path gives the maximum net complementary work. The earlier principle for work hardening solids [1] was restricted to this case; the work given in this paper is thus a consistent extension.

Secondly, consider a state of stress in which only one stress component and its associated strain component is non-zero. Uniaxial tension is an example. It is convenient in this case to plot the stress–strain relation (Fig. 7). We assume a conventional work hardening material: initially the behavior is elastic, followed by an inelastic range where

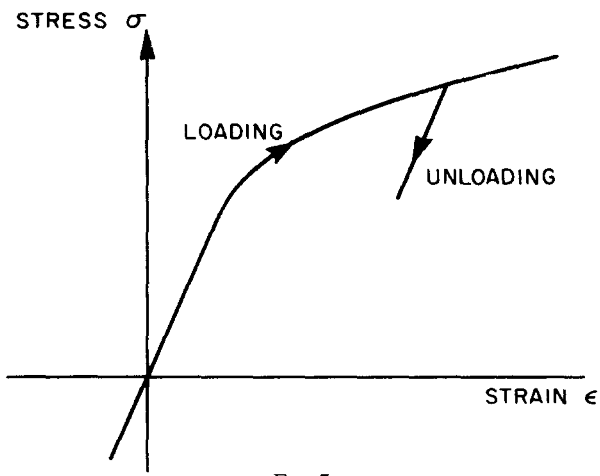

FIG. 7.

unloading occurs with the elastic slope. In this context a straight line stress path is a monotonic change from one stress state to another.

Provided that the material satisfies (1) and (2), it may be seen by inspection that the maximum complementary work occurs for a straight line path (as opposed to a path involving a change in sign of the stress rate) for a change in stress from σ^a to σ^b, no matter what the previous history. This alone is an important case, for many structures are analyzed on the assumption that only one generalized strain component is non-zero.

For more general states of stress it appears difficult to establish maximum complementary work paths for changes from any σ_{ij}^a to any σ_{ij}^b because of the effect of previous stress history. However, if we limit σ_{ij}^a to be identically zero and suppose that the material is in its virgin state, the p path can be established for simple materials. Consider for example, an isotropically hardening solid in its virgin state. It is clear that in this case a straight line (or radial) path from the origin (Fig. 8) is a path of maximum net complementary work. Here use of radial loading paths is tantamount to using a deformation theory of plasticity rather than an incremental theory.

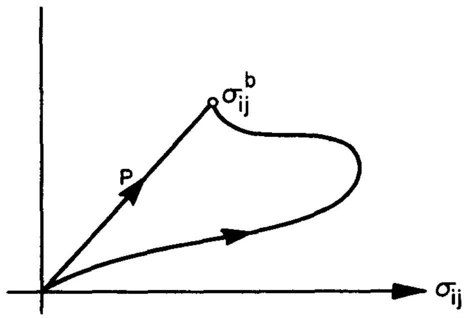

FIG. 8.

5. APPLICATION TO DYNAMIC LOADING

We shall now show that the general principle (16) may be usefully applied to the computation of safe displacement bounds for impulsive loading problems.

Suppose that the continuum has density ρ. The specific problem is as follows:

(i) at time $t < t_0$ the body is in its virgin state ($\sigma_{ij} = \varepsilon_{ij} = 0$).
(ii) at time $t = t_0$ the body acquires velocity $\dot{u}_i^0$.
(iii) for times $t > t_0$ the surface tractions are prescribed zero over part of the surface A_T, and the displacements are prescribed zero over the remainder of the surface A_u. Thus external forces do no work on the body.

At time $t > t_0$ suppose that the displacements, velocities and strains are given by u_i^t, $\dot{u}_i^t$, ε_{ij}^t. We may then write an energy balance equation relating the states at times t_0 and t:

$$\int_V \frac{\rho}{2}\dot{u}_i^0\dot{u}_i^0 \, dV = \int_V \frac{\rho}{2}\dot{u}_i^t\dot{u}_i^t \, dV + \int_V dV \int_0^{\varepsilon_{ij}^t} \sigma_{ij} \, d\varepsilon_{ij}. \tag{21}$$

The work term on the right hand side of (21) must in general be integrated over an inelastic path. It may be seen that u_i^t, ε_{ij}^t are compatible and satisfy all the conditions imposed on u_i^c, ε_{ij}^c, in equation (16). We may therefore substitute (21) into (16), with $\sigma_{ij}^a = 0$. This gives

$$\int_V \frac{\rho}{2}\dot{u}_i^0\dot{u}_i^0\,\mathrm{d}V - \int_V \frac{\rho}{2}\dot{u}_i^t\dot{u}_i^t\,\mathrm{d}V + \int_V \mathrm{d}V\int_0^{\sigma_{ij}^s} \varepsilon_{ij}^p\,\mathrm{d}\sigma_{ij}^p \geq \int_A T_i^s u_i^t\,\mathrm{d}A. \tag{22}$$

In addition

$$\int_V \frac{\rho}{2}\dot{u}_i^t\dot{u}_i^t\,\mathrm{d}V \geq 0. \tag{23}$$

Hence

$$\int_V \frac{\rho}{2}\dot{u}_i^0\dot{u}_i^0\,\mathrm{d}V + \int_V \mathrm{d}V\int_0^{\sigma_{ij}^s} \varepsilon_{ij}^p\,\mathrm{d}\sigma_{ij}^p \geq \int_A T_i^s u_i^t\,\mathrm{d}A \tag{24}$$

T_i^s, σ_{ij}^s remain any set of surface tractions and stresses which are in internal and external equilibrium, and are completely independent of the dynamic problem. The complementary work term as computed on the assumption that σ_{ij}^s is reached from the virgin state by a maximum complementary work path. Should it be possible that σ_{ij}^s may be reached by a completely reversible path at all points in the body, (24) reduces to a form given earlier [1]. By a suitable choice of T_i^s, therefore, bound on certain properties of the displacement field u_i^t may be computed in terms of the known initial energy of the dynamic problem and the complementary work of the assumed stress system. It must be emphasized that the equilibrium (*s*) system and the compatible (*c* or *t*) system satisfy different field equations, and therefore the left and right hand sides of (24) cannot be made arbitrarily close. The computation of bounds will be illustrated in the following section. It will be seen that the choice of T_i^s is dictated entirely by those properties of the displacement field u_i^t for which a bound is desired. We shall thus make deliberate use of the fact that T_i^s and u_i^t may be completely independent.

Under certain circumstances body forces (excluding inertia forces) may be included in the discussion. They have been omitted here for clarity, but were included in the earlier paper [1].

It should also be noted that the entire preceding discussion may be carried out for one and two dimensional continua by considering the appropriate generalized stresses and generalized strains. Two simple one dimensional continua will be discussed in the following section.

6. EXAMPLES

The examples which follow are intended only to illustrate the bound computation for assumed initial conditions. No exact solutions are given.

Consider first a uniform simply supported beam (Fig. 9) of span l, mass m per unit length, subjected to some initial velocity distribution $v(x)$ as shown. The material will be assumed to exhibit a linear work hardening moment curvature relation as shown in Fig. 10. The yield bending moment will be assumed to be M_0, the flexural rigidity in the

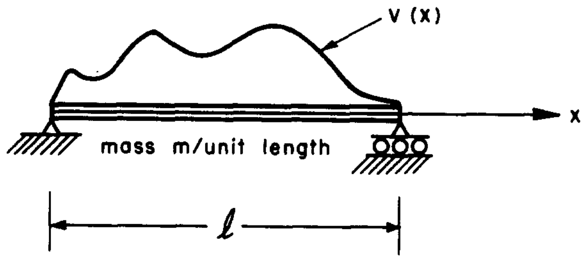

FIG. 9.

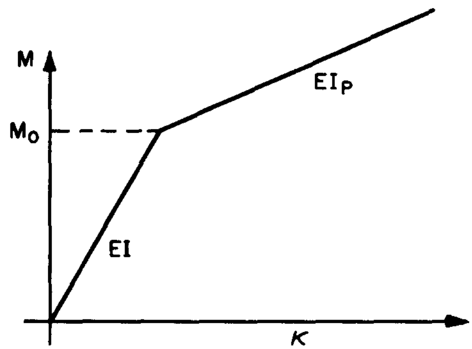

FIG. 10.

elastic range EI, and the tangent modulus in the plastic range to be EI_p. Unloading will occur with the elastic modulus. Further, shear strains will be assumed to be zero. The material satisfies the stability postulate and as noted in the previous section, a monotonic change in bending moment from one value to another will be a maximum net complementary work path. We shall assume that the beam is unstressed and undeformed before the impulse is applied, and shall therefore be concerned with changes from $M_0 = 0$. For a monotonic increase in moment from the origin, the elastic and plastic components of the curvature $\varkappa$ are respectively.

$$\varkappa^{el} = \frac{M}{EI}$$
$$\varkappa^{pl} = \frac{1}{EI_p} - \frac{1}{EI}\,(M - M_0) \qquad \text{for } M > M_0. \tag{25}$$

The complementary work function at any point is given by

$$\int_0^M \varkappa\,\mathrm{d}M = \int_0^M \varkappa^{el}\,\mathrm{d}M + \int_0^M \varkappa^{pl}\,\mathrm{d}M$$
$$= \frac{M^2}{2EI} + \frac{1}{2EI}\left(\frac{EI}{EI_p} - 1\right)(M - M_0)^2 \tag{26}$$

where the second term is zero if $M < M_0$.

A bound on the central displacement δ_c of the beam will be found. The static system chosen is that shown in Fig. 11, together with the statically determinate bending moment distribution. The analogous one-dimensional form of the bound expression (24) may now be written. It may be observed that the bound is not sensitive to the actual initial velocity distribution, but depends only on the total initial energy. In order to avoid specifying $v(x)$, therefore we shall put

$$K_0 = \int_0^l \frac{m}{2} v^2(x)\,dx. \tag{27}$$

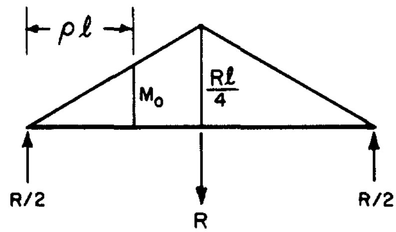

FIG. 11.

Further, let the complementary work term be C_s. (24) becomes, in this case

$$R\delta_c \leq K_0 + C_s. \tag{28}$$

It remains now to determine C_s. The bending moment distribution, superposed on Fig. 11, is broken into two regions: a region where $M \leq M_0$ and one where $M \geq M_0$. The plastic region ($M > M_0$) will vanish if

$$\frac{Rl}{4} \leq M_0. \tag{29}$$

The parameter ρ (Fig. 11) defining the length of the elastic region ($M < M_0$) for $R > 4M_0/l$ is given by

$$\rho l = \frac{2M_0}{R}. \tag{30}$$

Hence, using (26), the complementary work found for a p path is

$$\begin{aligned} C^s &= \frac{2}{2EI}\int_0^{l/2}\left(\frac{Rx}{2}\right)^2 dx + \frac{2}{2EI}\left[\frac{EI}{EI_p}-1\right]\int_{\rho l}^{l/2}\left(\frac{Rx}{2}-M_0\right)^2 dx \\ &= \frac{R^2l^3}{96EI} + \frac{3}{3EIR}\left[\frac{EI}{EI_p}-1\right]\left[\frac{Rl}{4}-M_0\right]^3 \end{aligned} \tag{31}$$

where the second term is zero for $R \leq 4M_0/l$. Substituting (31) into (28),

$$\delta_c \leq \frac{K_0 + \dfrac{R^2l^3}{96EI} + \dfrac{2}{3EIR}\left[\dfrac{EI}{EI_p}-1\right]\left[\dfrac{R}{4}-M_0\right]^3}{R}. \tag{32}$$

Any value of R may be substituted into the right hand side of (32) to obtain a bound on δ_c. It is more convenient, in this case, to recast (32) into dimensionless terms and to attempt to find an optimum value of R. (32) may be written

$$\frac{M_0\delta_c}{K_0 l} \leq \frac{1+\left(\frac{Rl}{M_0}\right)^2\left(\frac{M_0^2 l}{96EIK_0}\right)+64\left[\frac{EI}{EI_p}-1\right]\left(\frac{M_0}{Rl}\right)\left(\frac{M_0^2 l}{96EI}\right)\left(\frac{Rl}{4M_0}-1\right)^3}{\left(\frac{Rl}{M_0}\right)}. \tag{33}$$

In finding the optimum value of (Rl/M_0), it is simplest to consider separately the case $(Rl/M_0) \leq 4$. It may easily be shown that for this case the optimum bound is given by

$$\frac{Rl}{M_0} = \sqrt{\left(\frac{96EIK_0}{M_0^2 l}\right)}. \tag{34}$$

Putting

$$s = \frac{96EIK_0}{M_0^2 l}$$

(33) becomes

$$\frac{M_0\delta_c}{K_0 l} \leq 2\sqrt{\frac{1}{s}} \qquad \text{for } s \leq 16. \tag{35}$$

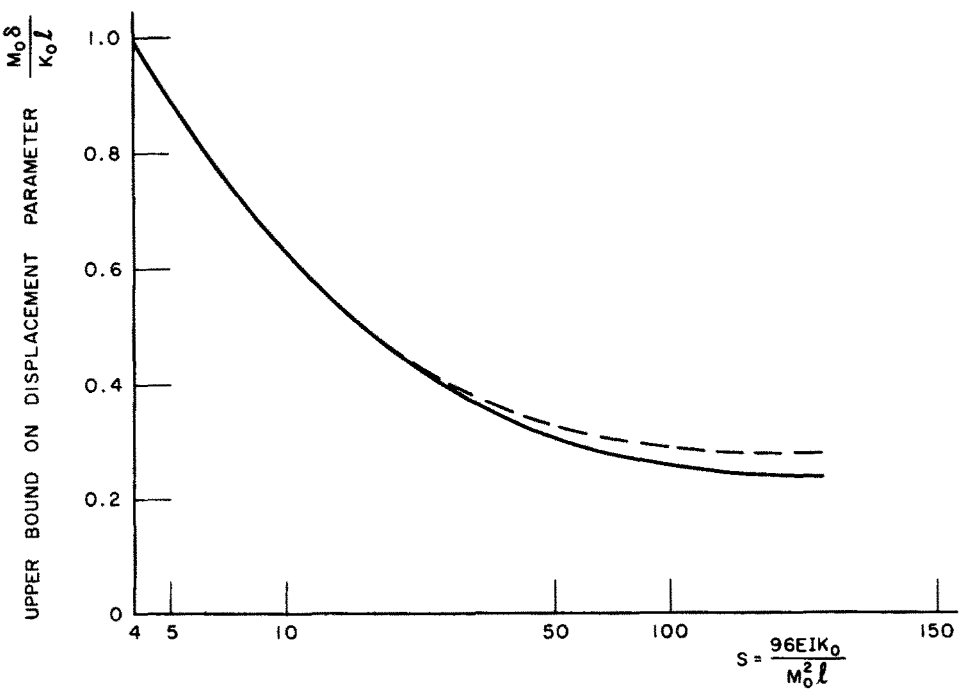

FIG. 12. Bound on central displacement for simply supported beam.

The optimization cannot be carried out in closed form for $(Rl/M_0) > 4$. It may be shown that the optimum value of Rl/M_0 is given by the appropriate root of the following equation:

$$\left(\frac{EI}{EI_p}\right)\left(\frac{Rl}{M_0}\right)^3-\frac{3}{4}\left(\frac{Rl}{M_0}\right)+2\left[\frac{EI}{EI_p}-1\right]=\frac{s}{64}\left(\frac{Rl}{M_0}\right). \tag{36}$$

For illustrative purposes, EI/EI_p was taken to be 10, and (36) was solved for various values of s. The solution for Rl/M_0 may then be substituted back into (33) to give the bound. The results of this calculation have been plotted on Fig. 12, giving the optimum bound as a function of s.

It is also of interest to compare this result with the more limited expression given in Reference [1]. This earlier expression has the same form, but restricts the bending moments in the static system (Fig. 11) to the elastic range, and thus requires $Rl/M_0 \leq 4$. Thus, for $s \geq 16$, (Rl/M_0) is taken equal to 4, and we obtain

$$\frac{M_0\delta}{K_0 l} \leq \frac{1}{4}+\frac{4}{s}. \tag{37}$$

(37) has been plotted as a dash line on Fig. 12. The difference between this line and the

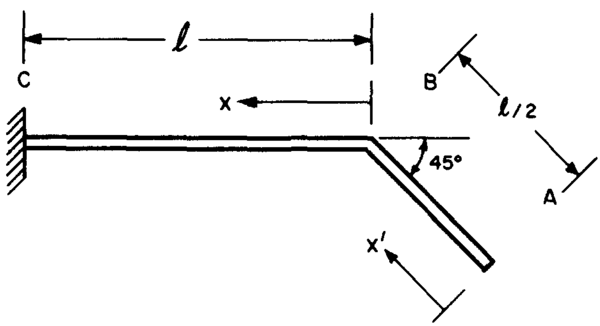

FIG. 13.

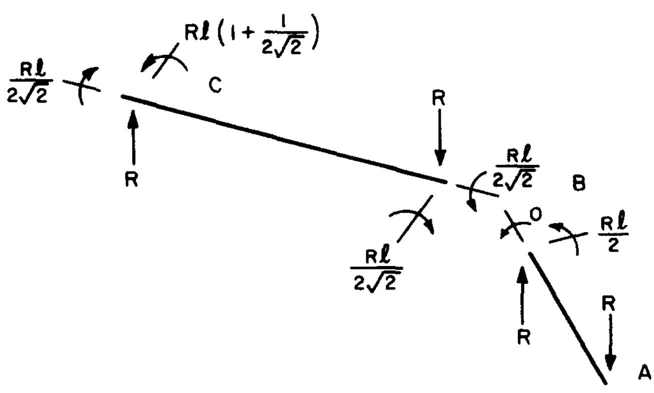

FIG. 14.

full line is distinct, although small (due to the limited work hardening defined by $EI/EI_p = 10$). For more strongly hardening materials, it is clear that the bound given in this paper will be much better than (37).

The second example we shall consider will be the uniform bent cantilever shown in plan in Fig. 13. As before let the total kinetic energy of the initial disturbance be K_0. We shall compute a bound on the tip displacement component δ transverse to the plane of the cantilever. For this purpose the appropriate static system is shown in Fig. 14, together with the distribution of bending moment and torsion moment.

We shall assume that the material is in its virgin state, that it work hardens isotropically and that shear strains are zero. Further we shall require that the constitutive relation should reduce to that used for the previous example for pure bending.

If the generalized stresses on an element of a one-dimensional continuum are represented by Q_j, and the associated plastic strains by q_j^{pl}, a general work hardening incremental stress–plastic strain relation for loading can be written (see, for example, Drucker [4])

$$\dot{q}_j^{pl} = G\frac{\partial\phi}{\partial Q_j}\frac{\partial\phi}{\partial Q_0}\mathrm{d}Q_k \tag{38}$$

where ϕ is a function of the generalized stresses, and G is a scalar function which is independent of the stress increment $\mathrm{d}Q_k$. In our case we shall take

$$\phi = M^2+T^2 \tag{39}$$

and assume that in the virgin state first yield occurs for

$$\phi = M_0^2. \tag{40}$$

(39) and (40) give a special case of a commonly used yield surface for moment and torque (Hodge [5]). Then, if $\varkappa$ is the curvature and ψ is the twist per unit length, (38) becomes

$$\begin{aligned} \mathrm{d}\varkappa^{pl} &= G(2M)(2M\,\mathrm{d}M+2T\mathrm{d}T) \\ \mathrm{d}\psi^{pl} &= G(2T)(2M\,\mathrm{d}M+2T\,\mathrm{d}T). \end{aligned} \tag{41}$$

In order that this relation should reduce to linear hardening for pure bending we shall assume that

$$G = \frac{G^*}{\phi^2} = \frac{G^*}{M^2+T^2} \tag{42}$$

where G^* is a constant. (41) then becomes

$$\begin{aligned} \mathrm{d}\varkappa^{pl} &= \frac{4G^*M}{M^2+T^2}(M\,\mathrm{d}M+T\,\mathrm{d}T) \\ \mathrm{d}\psi^{pl} &= \frac{4G^*T}{M^2+T^2}(M\,\mathrm{d}M+T\,\mathrm{d}T). \end{aligned} \tag{43}$$

Loading can be conveniently defined as $\dot{\phi} \geq 0$ i.e. a loading path does not decrease ϕ. We shall assume that elastic strains are given simply by

$$\varkappa^{el} = \frac{M}{EI} \qquad \psi^{el} = \frac{T}{GJ}. \tag{44}$$

The maximum complementary work path from the origin is a radial path, and it is therefore convenient to write a stress–total strain relation for such a path. Consider the radial loading path

$$T = kM, \qquad \dot{M} > 0. \tag{45}$$

From (39) and (40), first yield, and consequently first plastic strain, will occur for

$$M_y = \frac{M_0}{\sqrt{(1+k^2)}} \qquad T_y = \frac{kM_0}{\sqrt{(1+k^2)}} \tag{46}$$

Hence, from (44) and (43) the moment curvature relation is given by

$$\begin{aligned} \varkappa &= \varkappa^{el} + \varkappa^{pl} \\ &= \frac{M}{EI} + \int_{M_0/\sqrt{(1+k^2)}}^{M} 4G^* \frac{M^2(1+k^2)}{M^2(1+k^2)} \mathrm{d}M \\ &= \frac{M}{EI} + 4G^*\left(M - \frac{M_0}{\sqrt{(1+k^2)}}\right). \end{aligned} \tag{47a}$$

Similarly

$$\psi = \frac{T}{GJ} + 4G^*\left(T - \frac{M_0}{\sqrt{(1+k^2)}}\right). \tag{47b}$$

In each case the second term is zero if $M < M_0/1+k^2$. It may readily be shown that 47(a) will reduce to (25) provided that

$$4G^* = \left(\frac{1}{EI_p} - \frac{1}{EI}\right). \tag{48}$$

From (47) the complementary energy density is given by

$$\begin{aligned} \int q_j \,\mathrm{d}Q_j &= \int_0^M \varkappa \,\mathrm{d}M + \int_0^T \psi \,\mathrm{d}T \\ &= \frac{M^2}{2EI} + \frac{4G^*}{2}\left(M - \frac{M_0}{\sqrt{(1+k^2)}}\right) + \frac{T^2}{2GJ} + \frac{4G^*}{2}\left(T - \frac{kM_0}{\sqrt{(1+k^2)}}\right). \end{aligned} \tag{49}$$

Referring to Fig. 14, it is seen that the static system will be entirely elastic provided that yield does not occur at A; i.e. if

$$R^2 l^2 \left(1 + \frac{1}{2\sqrt{2}}\right)^2 + \frac{R^2 l^2}{(2\sqrt{2})^2} \leq M_0^2$$

i.e.

$$\frac{Rl}{M_0} \leq \frac{2}{\sqrt{(5+2\sqrt{2})}} \tag{50}$$

It will again be convenient to consider the purely elastic system separately. Taking $EI = GJ$ for the purposes of this example, and measuring x and x' as in Fig. 14, the complementary energy is given by

$$C^s = \int_L \frac{M^2}{2EI}\,ds + \int_L \frac{T^2}{2EI}\,ds$$
$$= \int_0^{l/2} \frac{(Rx')^2}{2EI}\,dx' + \int_0^l \left(\frac{Rl}{2\sqrt{2}} + Rx\right)^2 \frac{dx}{2EI} + \int_0^l \left(\frac{Rl}{2\sqrt{2}}\right)^2 \frac{dx}{2EI}$$
$$= \frac{R^2 l^3}{16EI}(5+2\sqrt{2}) \qquad \text{for} \qquad \frac{Rl}{M_0} \leq \frac{2}{\sqrt{(5+2\sqrt{2})}}. \tag{51}$$

Thus, in this case, the bound principle (24) becomes

$$R\delta \leq K_0 + \frac{R^2 l^3}{16EI}(5+2\sqrt{2}). \tag{52}$$

It may readily be shown that the optimum bound is given for

$$R = \sqrt{\left(\frac{16EIK_0}{(5+2\sqrt{2})l^3}\right)} \tag{53}$$

giving

$$\frac{M_0\delta}{K_0 l} \leq \sqrt{\left(\frac{5+2\sqrt{2}}{s}\right)}, \qquad s \leq 1 \tag{54a}$$

where

$$s = \frac{4EIK_0}{M_0^2 l}. \tag{54b}$$

For $Rl/M_0 \geq 2/\sqrt{(5+2\sqrt{2})}$ an elastic–plastic boundary will occur in the beam. It may be seen that the physical parameters of the problem are such that there is only one such boundary which moves away from C as R is increased. We shall consider only values of R such that the boundary occurs in BC: let the boundary be distance ρl from B. The initial yield equation (40) is satisfied here, hence

$$R^2\left(\frac{l}{2\sqrt{2}} - \rho l\right)^2 + R^2\left(\frac{l}{2\sqrt{2}}\right)^2 = M_0^2$$

or

$$\left(\frac{Rl}{M_0}\right)^2 = \frac{1}{\rho^2 + \frac{1}{2}\rho + \frac{1}{4}}. \tag{55}$$

We have limited ρ to lie between 0 and 1, and hence will consider

$$\frac{2}{\sqrt{(5+\sqrt{2})}} \leq \frac{Rl}{M_0} \leq 4. \tag{56}$$

The moment–torque ratio in the plastic region is a function of x, and is given by

$$k = \frac{T}{M} = \frac{\dfrac{Rl}{2\sqrt{2}}}{R\left[\dfrac{l}{2\sqrt{2}} + x\right]} = \frac{1}{1 + 2(\sqrt{2})\dfrac{x}{l}}. \tag{57}$$

To account for the plastic contribution to the complementary work the following terms, from (49), must be added to (51) for the range of Rl/M_0 given in (56):

$$C^{sp} = \int_{\rho l}^{l} \frac{4G^*}{2}\left(Rx - \frac{M_0\left[1+2(\sqrt{2})\frac{x}{l}\right]}{\sqrt{\left[2+4(\sqrt{2})\frac{x}{l}+8\frac{x^2}{2}\right]}}\right)\mathrm{d}x + \int_{\rho l}^{l} \frac{4G^*}{2}\left(\frac{Rl}{2\sqrt{2}} - \frac{M_0}{\sqrt{\left[2+4(\sqrt{2})\frac{x}{l}+8\frac{x}{l^2}\right]}}\right)^2 \mathrm{d}x$$

$$= \frac{4G^*}{2}R^2l^3\left[\frac{1}{3}\left(\frac{x}{l}\right)^3 + \left\{\frac{1}{8}+\left(\frac{M_0}{Rl}\right)^2\right\}\left(\frac{x}{l}\right) - \frac{1}{8}\left(\frac{M_0}{Rl}\right)\left[2(\sqrt{2})\frac{x}{l}-1\right]\sqrt{\left(2+4\;2\frac{x}{l}+8\frac{x^2}{l^2}\right)}\right.$$

$$\left. + \frac{1}{8}\left(\frac{M_0}{Rl}\right)\left[2(\sqrt{2})\frac{x}{l}-3\right] \text{arc sinh}\left[1+2(\sqrt{2})\frac{x}{l}\right]\right]_{\rho l}^{l}. \qquad (58)$$

The bound expression (24) thus gives

$$R\delta \leq K_0 + \frac{R^2l^3}{16EI}(5+2\sqrt{2}) + C^{sp}. \qquad (59)$$

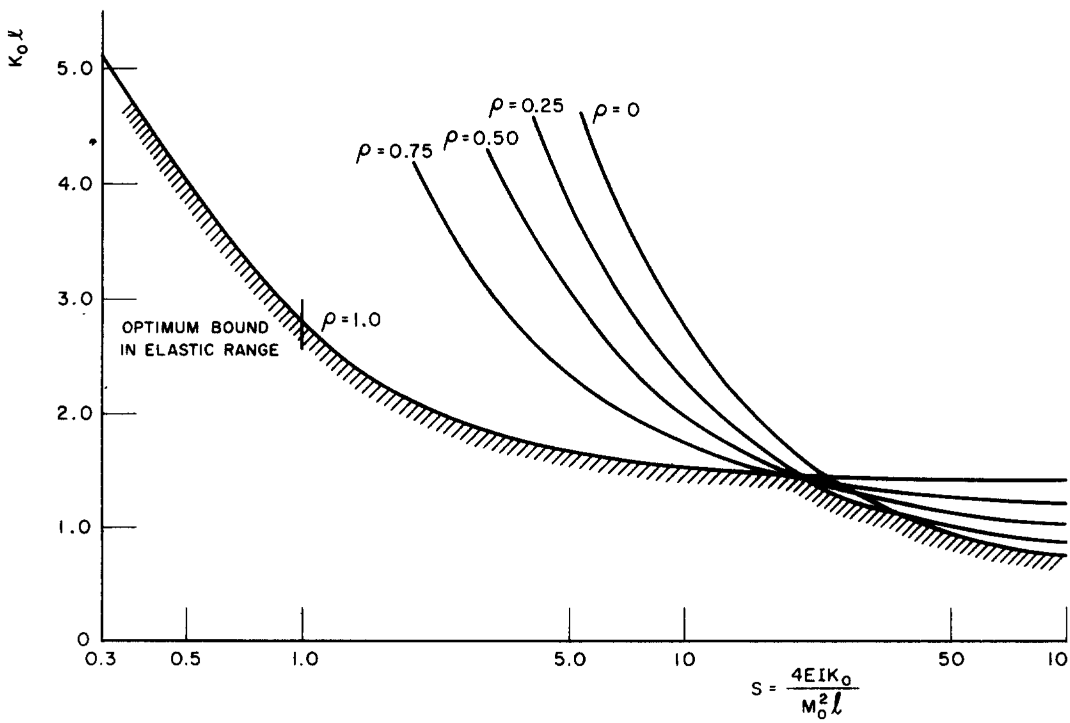

FIG. 15. Bound on tip displacement for bent cantilever.

In dimensionless terms

$$\frac{M_0\delta}{K_0} \le \frac{1+\frac{5+2\sqrt{2}}{4}\left(\frac{Rl}{M_0}\right)^2\left(\frac{M_0^2 l}{4EIK_0}\right)+4\left(\frac{EIC^{sp}}{R^2 l^3}\right)\left(\frac{Rl}{M_0}\right)^2\left(\frac{M_0^2 l}{4EIK_0}\right)}{\left(\frac{Rl}{M_0}\right)}. \tag{60}$$

There appears to be no possibility of finding an analytical expression for the optimum bound. In consequence the bound has been evaluated numerically as a function of s for values of R corresponding to $\rho = 1{\cdot}0$, 0·75, 0·50, 0·25 and 0. These curves are shown, together with (53) in Fig. 15. The envelope to the family of curves shown is an upper bound on the displacement for varying s. In computing these curves, the value $EI/EI_p = 5$ was taken.

7. CONCLUSIONS

To the knowledge of the author, no analytical solutions for impulsively loaded work hardening structures or continua have been obtained. Some numerical results have been obtained; for example, Witmer *et al.* [6]. In these cases, however, large deformations have been included and hence no direct comparison with the present technique is available.

The bound technique is relatively simple to apply, even in comparatively complex structures and continua. Its utility will depend largely on its accuracy. Consideration of accuracy must be divided into two parts. First, how close is the bound to the analytical solution it approximates, and second, how good is the analytical solution? The second factor is beyond the scope of this paper, since it depends on how well the mathematical description fits the physical problem. The most important restriction is the neglect of large geometry changes: the analytical solution can be valid only for small impulse magnitudes.

The first factor can be evaluated only by considering a large number of examples, and comparing the bounds to full solutions. Previous experience with such methods [1], [7], [8] show that at best the bound can lie within a few per cent of the actual answer, and at worst will give at least the order of magnitude of the displacement. It would appear, therefore, that the method may be significant as a preliminary design procedure in complex configurations.

Acknowledgment—The author is indebted to Professor D. C. Drucker for his assistance and criticism in the preparation of this paper.

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(*Final draft received* 4 *June* 1965)

Résumé—Certaines études précédentes présentant une technique pour les déplacements à rebondissement suivant une charge impulsive dans un continuum non élastique, ont été prolongées pour être appuyées par de plus puissants résultats sur les matières stable durcissantes à travail. Ce supplément, consistant d'un exposé évident en lui-même, sur le comportement de la matière, est ajouté au postulat de stabilité de Drucker pour les solides indépendants du temps et dépendants de la trajectoire.

Zusammenfassung—Frühere Arbeiten über ein Verfahren für beschränkende Verschiebungen in Folge impulsiver Belastung unelastischer Kontinua werden noch ergänzt um überzeugendere Ergebnisse mit stabilen, durch Bearbeitung erhärtende Materialien anzugeben. Dieser Ergänzung wird noch die offensichtliche Erklärung hinzugefügt betreffend das Materialverhalten zur Drucker's Stabilitätshypothese für zeitunabhängige jedoch weglängen abhängige Festkörper.

Абстракт—Раньше опубликованный метод ограничения перемещений, вытекающих из импульсивной нагрузки в неупругом континууме, разработан в настоящем труде с целью получения более точных результатов для устойчивых затвердевающих тел. Это расширение метода достигнуто посредством включения самоочевидного предложения о поведении тел по постулату Дрюкера относительно устойчивости независимых от времени, зависимых от пути, твердых тел.